

## Differentiating Inverse Hyperbolic Functions

### Example

Differentiate  $y = \sinh^{-1} x$

$$\begin{aligned}
 & y = \sinh^{-1} x \\
 \Rightarrow & \sinh y = x \\
 \Rightarrow & \cosh y \frac{dy}{dx} = 1 \\
 \Rightarrow & \frac{dy}{dx} = \frac{1}{\cosh y} \\
 & = \frac{1}{\sqrt{1 + \sinh^2 y}} \\
 & = \frac{1}{\sqrt{1 + x^2}}
 \end{aligned}$$

Alternatively,

$$\begin{aligned}
 & y = \sinh^{-1} x \\
 & = \ln(x + \sqrt{1 + x^2}) \\
 \Rightarrow & \frac{dy}{dx} = \frac{1}{x + \sqrt{1 + x^2}} \left( 1 + \frac{1}{2}(1 + x^2)^{-\frac{1}{2}} 2x \right) \\
 & = \frac{1}{x + \sqrt{1 + x^2}} \left( 1 + \frac{x}{\sqrt{1 + x^2}} \right) \\
 & = \frac{1}{x + \sqrt{1 + x^2}} \frac{\sqrt{1 + x^2} + x}{\sqrt{1 + x^2}} \\
 & = \frac{1}{\sqrt{1 + x^2}}
 \end{aligned}$$

### Example

Integrate  $\int \frac{1}{\sqrt{a^2 + x^2}} dx$

$$x = a \sinh u, \frac{dx}{du} = a \cosh u$$

$$\begin{aligned}
 \int \frac{1}{\sqrt{a^2 + x^2}} dx &= \int \frac{a \cosh u}{\sqrt{a^2 + a^2 \sinh^2 u}} du \\
 &= \int \frac{a \cosh u}{a \cosh u} du \\
 &= \int 1 du \\
 &= u + C \\
 &= \sinh^{-1} \left( \frac{x}{a} \right) + C
 \end{aligned}$$

**Example**Differentiate  $y = \cosh^{-1} x$ 

$$\begin{aligned}
 & y = \cosh^{-1} x \\
 \Rightarrow & \cosh y = x \\
 \Rightarrow & \sinh y \frac{dy}{dx} = 1 \\
 \Rightarrow & \frac{dy}{dx} = \frac{1}{\sinh y} \\
 & = \frac{1}{\sqrt{\cosh^2 y - 1}} \\
 & = \frac{1}{\sqrt{x^2 - 1}}
 \end{aligned}$$

*Alternatively,*

$$\begin{aligned}
 & y = \cosh^{-1} x \\
 & = \ln(x + \sqrt{x^2 - 1}) \\
 \Rightarrow & \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 - 1}} \left( 1 + \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}} 2x \right) \\
 & = \frac{1}{x + \sqrt{x^2 - 1}} \left( 1 + \frac{x}{\sqrt{x^2 - 1}} \right) \\
 & = \frac{1}{x + \sqrt{x^2 - 1}} \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} \\
 & = \frac{1}{\sqrt{x^2 - 1}}
 \end{aligned}$$

**Example**Integrate  $\int \frac{1}{\sqrt{x^2 - a^2}} dx$ 

$$x = a \cosh u, \quad \frac{dx}{du} = a \sinh u$$

$$\begin{aligned}
 \int \frac{1}{\sqrt{x^2 - a^2}} dx &= \int \frac{a \sinh u}{\sqrt{a^2 \cosh^2 u - a^2}} du \\
 &= \int \frac{a \sinh u}{a \sinh u} du \\
 &= \int 1 du \\
 &= u + C \\
 &= \cosh^{-1} \left( \frac{x}{a} \right) + C
 \end{aligned}$$

**Example**Differentiate  $y = \tanh^{-1} x$ 

$$\begin{aligned}
 & y = \tanh^{-1} x \\
 \Rightarrow & \tanh y = x \\
 \Rightarrow & \operatorname{sech}^2 y \frac{dy}{dx} = 1 \\
 \Rightarrow & \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} \\
 & = \frac{1}{1 - \tanh^2 y} \\
 & = \frac{1}{1 - x^2}
 \end{aligned}$$

**Example**Integrate  $\int \frac{1}{a^2 - x^2} dx$ 

$$\begin{aligned}
 x = a \tanh u, \frac{dx}{du} = a \operatorname{sech}^2 u & \qquad \int \frac{1}{a^2 - x^2} dx = \int \frac{a \operatorname{sech}^2 u}{a^2 - a^2 \tanh^2 u} du \\
 & = \int \frac{1}{a} du \\
 & = \frac{u}{a} + C \\
 & = \frac{1}{a} \tanh^{-1} \left( \frac{x}{a} \right) + C
 \end{aligned}$$

*Alternatively*

$$\begin{aligned}
 \int \frac{1}{a^2 - x^2} dx &= \frac{1}{2a} \int \left( \frac{1}{a+x} + \frac{1}{a-x} \right) dx \\
 &= \frac{1}{2a} (\ln(a+x) - \ln(a-x)) + C \\
 &= \frac{1}{2a} \ln \left( \frac{a+x}{a-x} \right) + C
 \end{aligned}$$

**Example**

Evaluate  $\int_0^4 \frac{1}{\sqrt{9+x^2}} dx$

$$\begin{aligned}\int_0^4 \frac{1}{\sqrt{9+x^2}} dx &= \left[ \sinh^{-1} \left( \frac{x}{3} \right) \right]_0^4 \\ &= \sinh^{-1} \left( \frac{4}{3} \right) \\ &= \ln \left( \frac{4}{3} + \sqrt{1 + \left( \frac{4}{3} \right)^2} \right) \\ &= \ln \left( \frac{4}{3} + \frac{5}{3} \right) \\ &= \ln 3\end{aligned}$$

**Example**

Integrate  $\int \frac{1}{\sqrt{x^2+x}} dx$

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+x}} dx &= \int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{2^2}}} dx \\ &= \cosh^{-1} \left( \frac{x + \frac{1}{2}}{\frac{1}{2}} \right) + C \\ &= \cosh^{-1} (2x + 1) + C\end{aligned}$$

**Example**

Find  $\int \frac{1}{7 \sinh x + 3 \cosh x - 9} dx$

*We could try and find a  $R \sinh(x + \alpha)$  expression for the denominator, but it is easier to convert everything to exponentials and substitute  $u = e^x$*

**Tip (Useful Substitutions)**

Some helpful substitutions for integrating tougher integrals:

Expression	Substitution
$a^2 + x^2$	$x = a \tan u$ or $x = a \sinh u$
$a^2 - x^2$	$x = a \sin u$
$x^2 - a^2$	$x = a \cosh u$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} dx = \sinh^{-1} \left( \frac{x}{a} \right) + C$$

$$\int \sqrt{a^2 + x^2} dx = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{a^2}{2} \sinh^{-1} \left( \frac{x}{a} \right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + C$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x}{a} \right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \left( \frac{x}{a} \right) + C$$

$$\int \sqrt{x^2 - a^2} dx = \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1} \left( \frac{x}{a} \right) + C$$

**Tip (Useful Integration by parts)**

Integrate by parts  $\frac{du}{dx} = 1, v = f(x)$  to get  $x \times f'(x)$  which we can integrate

$$\int \ln x dx = [x \ln x] - \int \left( x \cdot \frac{1}{x} \right) dx$$

$$\int \sin^{-1} x dx = [x \sin^{-1} x] - \int \left( x \cdot \frac{1}{\sqrt{1-x^2}} \right) dx$$

$$\int \cos^{-1} x dx = [x \cos^{-1} x] - \int \left( x \cdot \frac{-1}{\sqrt{1-x^2}} \right) dx$$

$$\int \tan^{-1} x dx = [x \tan^{-1} x] - \int \left( x \cdot \frac{1}{1+x^2} \right) dx$$

$$\int \sinh^{-1} x dx = [x \sinh^{-1} x] - \int \left( x \cdot \frac{1}{\sqrt{1+x^2}} \right) dx$$

$$\int \cosh^{-1} x dx = [x \cosh^{-1} x] - \int \left( x \cdot \frac{1}{\sqrt{x^2-1}} \right) dx$$

$$\int \tanh^{-1} x dx = [x \tanh^{-1} x] - \int \left( x \cdot \frac{1}{1-x^2} \right) dx$$

**Tip** (Integration by Parts to Remove a function)

$$\int x \cos x \, dx = [x \cos x] - \int 1 \cdot \sin x \, dx$$
$$\int x^n \cos x \, dx = [x^n \cos x] - \int nx^{n-1} \sin x \, dx$$
$$\int xe^x \, dx = [xe^x] - \int 1 \cdot e^x \, dx$$

**Tip**

Things you should always think about when integrating:

- Can I use partial fractions?
- Does the numerator look like the derivative of the denominator? [ $\ln(\cdot)$ ]
- Can I complete the square?
- Does it look like one of the substitutions above?
- Can I integrate by parts?